Notes on Synchrotron Radiation

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Version of May 18, 2007

Contents

1 Introduction

The purpose of these notes is twofold. First, it is an attempt to update the brief chapter on synchrotron radiation in the book by Mike Syphers and me[1]. Second, an immediate context for application of this material is the emittance transfer experiment under assembly at Fermilab[2].

When we were putting together the material for the book about twenty years ago, synchrotron radiation was mostly an irritation in the construction of accelerators for high energy physics. For electron synchrotrons, the main problem was the provision of enough RF acceleration to overcome the radiation loss per turn. Even in the case of proton synchrotrons of sufficiently high energy such as the LHC, radiation from protons was would be a major contributor to the cryogenic heat load and a vacuum system design problem.

Since then interest in synchrotron radiation has grown dramatically as the value of "light sources" for research and application in virtually every field of science and technology has been realized. A striking indicator of this circumstance is today's transition of such laboratories as DESY and SLAC away from emphasis on HEP toward fourth generation light sources.

Therefore, it is not enough to comment on the radiation loss per turn and the critical energy of the photons. So the discussion in the succeeding sections goes somewhat further, without attempting to substitute for the completeness of, say, Jackson's *Classical Electrodynamics*[3]. But, at a minimum, it is necessary to expand on single-particle radiation including deviation from a circle as in the edge effect and to discuss coherent radiation.

The experiment under assembly at Fermilab combines both the attractive and aggravating aspects of synchrotron radiation. Aggravating in the potential for coherent synchrotron radiation to conflict with the phase space interchange goal of the experiment; attractive in its benefits for beam diagnostics. It is remarkable to me that these effects have relevance at an electron energy of only 15 MeV. Mention of these effects in the experiment will appear throughout these notes, with the main discussion in the concluding sections.

2 Transit to the Lienard-Wiechert Potentials

In our book, Mike and I avoided the use of the vector potential, my recollection being that I had not seen it as an undergraduate. By now it's probably introduced in high school. Besides it would be artificial to attempt to address the subjects of these notes without the use of the retarded potentials.

2.1 Equations for the Potentials

Since $\nabla \cdot \vec{B} = 0$ the Helmholtz decomposition theorem of vector analysis[4] says that we can express the magnetic field in terms of derivatives of another vector, $\vec{A}$, according to

$$\vec{B} = \nabla \times \vec{A}. \tag{1}$$

Then, using $\vec{A}$, Faraday's Law, $\nabla \vec{E} = -\partial \vec{B}/\partial t$, Eq. 1 can be written

$$\nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0, \tag{2}$$

and since the curl vanishes, the rest of the Helmholtz theorem says that the quantity in parentheses can be written as the gradient of some scalar function, Φ. So now the electric field may be expressed as

$$\vec{E} = -\nabla\Phi - \frac{\partial\vec{A}}{\partial t}. \tag{3}$$

Eqs. 1, 3 are the standard expressions for the fields in terms of the vector and scalar potentials. The two of Maxwell's equations that are homogeneous in the fields were used to obtain them.

In vacuum form, the other two of Maxwell's equations are

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \tag{4}$$

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \tag{5}$$

where ρ and $\vec{j}$ are the charge and current densities respectively. Take Eqs. 1, 2, insert them into Eq. 5, and rearrange to give

$$\nabla\left(\nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right) - \nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \mu_0\vec{j} \tag{6}$$

where use has been made of the identity $\nabla \times \nabla \times \vec{A} = \nabla(\nabla \times \vec{A}) - \nabla^2\vec{A}$. The components of $\vec{A}$ are linked only by the first term on the left. It would be convenient if somehow it went away. That it could be eliminated was recognized in the latter part of the 19th century, and that it can be set to zero is called the Lorentz condition.[5].

In Eq. 1 if you replace $\vec{A}$ by $\vec{A} + \nabla\xi$ where ξ is some scalar function of space and time then nothing abot $\vec{B}$ changes because the curl of a gradient is zero. Similarly, the electric field is unchanced by the replacement $\Phi \to \Phi - \partial\xi/\partial t$. The condition that the parenthetical term in Eq. 6 be an invariant is that ξ satisfy the homogeneous wave equation. Then with application of the gradient operator, the entire first term in the equation vanishes. With a related though shorter argument applied to Eq. 4, the equations for the potentials become

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\frac{\rho}{\varepsilon_0} \tag{7}$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{j} \tag{8}$$

and the components of the potentials are no longer coupled.

2.2 Retarded Potentials

The four equations represented by Eqs. 7, 8 are all of the same form, so we need only solve one of them. The main thing to keep in mind during this context is the finite speed of light. If you are sitting at some point $\vec{r}$ at time t, then the signal from a field source located at $\vec{r}\,'$ must have been emitted at a time $t' = t - R/c$ earlier, where R is the magnitude of the distance between $\vec{r}$ and $\vec{r}\,'$.

This paragraph is based on Born and Wolf[6]. From electrostatics, one might expect that a trial solution for Φ would be

$$\Phi(\vec{r}, t) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}\,', t - R/c)}{R} dV' \tag{9}$$

and put this into Eq. 7 to verify that this works. There is a pole at $R = 0$, so set $\Phi = \phi_1 + \phi_2$ where ϕ_1 is the integral within a sphere of radius a and ϕ_2 is the integral from a out to infinity. For ϕ_2, differentiation under the integral sign is permitted and ϕ_2 is found to satisfy the homogeneous wave equation. For ϕ_1, look at the limit as a tends to zero. Then $\nabla^2\phi_1$ will approach the electrostatic potential of a homogeneously charged sphere:

$$\nabla^2\phi_1 = -\frac{1}{\varepsilon_0}\rho(\vec{r}, t). \tag{10}$$

Finally, for a sufficiently small

$$\ddot{\phi}_1 \approx \frac{1}{2}a^2\ddot{\rho} \tag{11}$$

and so tends to zero with a. Therefore Φ as represented by Eq. 9 is a solution of Eq. 7 as are, by extension, the solutions for the vector potential:

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{j}(\vec{r}\,', t - R/c)}{R} dV'. \tag{12}$$

Eqs. 9 and 12 are the retarded potentials.

To apply the retarded potentials to a moving "point" charge, I switch to the discussion in Panofsky and Phillips[7], though with changes in notation. Two factors have to be taken into consideration: the charge may be in motion and its structure should not appear in the result. Think of a sphere collapsing inward toward the point of observation located at $\vec{r}$. If the charge were at rest, the field source collected by a volume element of surface area dS and thickness dr would be $[\rho]dSdr$, where $[\rho]$ is the charge density at the retarded time. But if the charge is moving with velocity $\vec{v}$, then if there is a component of $\vec{v}$ directed toward the point of observation, then the quantity of charge in this volume element will be reduced by $[\rho]dS(\vec{v}\cdot\vec{n}dt$, where $\vec{n}$ is the unit vector from the source point to the point of observation. Therefore, the amount of charge included in this volume element is

$$de = [\rho]dV' - [\rho]\vec{v}\cdot\vec{n}dSdt. \tag{13}$$

Since $dt = dr/c$. we can solve for the retarded charge density:

$$[\rho]dV' = de\frac{1}{1-\vec{\beta}\cdot\vec{n}} \tag{14}$$

with $\vec{\beta} = \vec{v}/c$. Since in the point charge limit, the region occupied by the charge is negligible, Eq. 9 yields

$$\Phi = \frac{e}{4\pi\varepsilon_0 R}\left[\frac{1}{(1-\vec{n}\cdot\vec{\beta})}\right]_{ret}, \tag{15}$$

and the corresponding expression for the vector potential is

$$\vec{A} = \frac{\mu_0 e}{4\pi R}\left[\frac{\vec{v}}{(1-\vec{n}\cdot\vec{\beta})}\right]_{ret}. \tag{16}$$

These are the Lienard-Wiechert potentials. The subscript "*ret*" means that the velocity is to be evaluated at the retarded time.

The fields yielded by Eqs. 15 and 16 are

$$\vec{E} = \frac{e}{4\pi\varepsilon_0}\left[\frac{\vec{n}-\vec{\beta}}{\gamma^2(1-\vec{\beta}\cdot\vec{n})^3R^2}\right]_{ret} + \frac{e}{4\pi\varepsilon_0 c}\left[\frac{\vec{n}\times[(\vec{n}-\vec{\beta})\times d\vec{\beta}/dt]}{(1-\vec{\beta}\cdot\vec{n})^3R}\right]_{ret} \tag{17}$$

$$\vec{B} = \frac{1}{c}\vec{n}\times\vec{E} \tag{18}$$

and it is interesting to note that differentiation of the Lienard-Wiechert potentials to get the fields is not all that easy; here I am just reproducing the result in Jackson, with a change to SI units. The first term in Eq. 17 has the $1/R^2$ dependence associated with the field of a uniformly moving charge and the $1/R$ of the second term is characteristic of the radiation field resulting from acceleration. Both terms are necessary for discussion of intra-bunch processes; only the second is needed for the far field effects of the next sections.

3 Single Particle Synchrotron Radiation

3.1 The Larmor Formula

Suppose the speed of the particle is much less than that of light. Then the far field term in Eq. 17 is

$$\vec{E} = \frac{e}{4\pi\varepsilon_0 cR}[\vec{n} \times (\vec{n} \times d\vec{\beta}/dt)]_{ret} \tag{19}$$

and with Poynting's vector, $\vec{S} = \vec{E} \times \vec{B}/\mu_0$, this leads to the power per unit solid angle

$$\frac{dP}{d\Omega} = R^2|\vec{S}| = \frac{e^2}{(4\pi)^2\varepsilon_0}\frac{a^2}{c^3}\sin^2\theta \tag{20}$$

where a is the magnitude of the acceleration and θ is the "latitude" in a coordinate system with $\theta = 0$ in the direction of the acceleration. Integration over the solid angle $d\Omega = \sin\theta d\theta d\phi$ yields the familiar Larmor formula:

$$P = \frac{1}{6\pi\varepsilon_0}\frac{e^2a^2}{c^3} \tag{21}$$

The relativistic extension of Eq. 21 was reproduced in the book by Syphers and me[1] and need not be detailed here, other than to state that for centripetal acceleration a factor of γ^4 multiplies the Larmor formula. The associated radiation extends up to an angular frequency characterized by

$$\omega_c = \frac{3}{2}\gamma^3\frac{c}{\rho} \tag{22}$$

where ρ is the bend radius.

Some of the references that I'm looking at use Gaussian units, others SI. I would like to switch to expressing equations in a form independent of the set of units. After inclusion of the factor γ^4, divide Eq. 21 by $\hbar\omega_c$, set $a = c^2/\rho$, make use of Eq. 22, and multiply by the time of passage through a bend magnet of angle χ, $\delta t = \rho\chi/c$. The result is

$$N_c = \left(\frac{2}{3}\right)^2 \alpha\gamma\chi \tag{23}$$

where α is the fine structure constant. Eq. 23 represents the number of photons if they were all at the critical energy arising from a single passage through the bend.

At A0, one of the dogleg magnets produces a bend of 22.5 degrees. At 15 MeV, $N_c = 0.037$. The critical energy $\hbar\omega_c$ is 0.97×10^{-21} J or 6×10^{-3} eV. If the radiation from a 1 nC bunch were incoherent, the radiated energy would be 3×10^{-12} J, and would be difficult to detect. The critical wavelength, $\lambda_c = 2\pi c/\omega_c$, is about 125 μm.

3.2 Distribution in Energy and Angle

Jackson uses the variable I to denote the energy radiated during a single passage through the region of obsevation and $d^2I/(d\omega d\Omega)$ for the energy distribution in frequency and solid angle. Proceeding along the lines of the discussion of the Larmor formula above, Jackson's Eq. 14.83 on page 674 of the second edition can be expressed as

$$\frac{d^2I}{d\hbar\omega d\Omega} = \frac{3\alpha}{4\pi^2}\left(\frac{\omega}{\omega_c}\right)^2 \gamma^6 \left(\frac{1}{\gamma^2} + \theta^2\right)^2 \left[K_{2/3}^2(\xi) + \frac{\theta^2}{(1/\gamma^2) + \theta^2} K_{1/3}^2(\xi)\right] \tag{24}$$

where the parameter ξ is defined by

$$\xi \equiv \frac{\omega}{2\omega_c}\left(1 + \gamma^2\theta^2\right)^{3/2} \tag{25}$$

4 Coherent Synchrotron Radiation

In recent years, high density electron bunches have led to interest in coherent synchrotron radiation (CSR). Within a bunch, there is an instability that has been observed in compressors, for example[?]. Internal to the bunch it is not meaningful to distinguish near field from far field, and so simulation codes that treat this process recognize this circumstance[?]. For beam diagnostics, the far field remains the aspect for detection, and this we take a look at first, specifically in the case of the Fermilab experiment.

My initial reaction to Paul Emma's suggestion[?] that CSR may be of interest in this case was disbelief. How can synchrotron radiation be an issue at 15 MeV? But the coherent process varies as the square of the current, and after all this is how ordinary radio antennas work. At the conclusion of Sec. 3.1 I noted that the critical wavelength, λ_c, was about 190μm. The bunch length is only one order of magnitude longer, a far cry from the situation in even the earliest electron synchrotrons.

References

[1] D. A. Edwards and M. J. Syphers, "Introduction to the Physics of High Energy Accelerators", Wiley 1993.

[2] See for example P. Emma, Z. Huang, K.-J. Kim, P. Piot, "Transverse-to-Longitudinal Emittance Exchange to Improve Performance of High-Gain Free-Electron Lasers", FNAL PUB-06-256-AD, June 2006, and references contained therein.

[3] J. D. Jackson, "Classical Electrodynamics", Second Edition, Wiley 1975.

[4] I was not able to find the original reference. Possibly it can be found in his many papers of the 1870s.

[5] A discussion of the origin of gauge invariance may be found in Jackson and Okun, Rev. Mod. Phys. *73*, 663 (2001).

[6] M. Born and E. Wolf, "Principles of Optics", 3rd Edition, Pergamon Press, 1965

[7] W. Panofsky and M. Phillips, "Classical Electricity and Magnetism", Addison-Wesley, 1955